A Thought on Approximation by Bi-Analytic Functions

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Dedicated to the memory of André Boivin, a kind and gentle friend.

Abstract. A different approach to the problem of uniform approximations by the module of bi-analytic functions is outlined. This note follows the ideas from [8, 10, 9, 11, 6] and the more recent paper [1], regarding approximation of $\bar{z}$ by analytic functions.

1 Introduction

The ideas sketched in this note were inspired by the talk of J. Verdera at the Approximation Theory Conference dedicated to A. Boivin held at the Fields Institute in Toronto in July 2016 . Denote by $R_2(K)$ the uniformly closed rational module generated by functions $f(\zeta) + \bar{z}g(\zeta)$, with f and g analytic in the neighborhood of a compact set K in $\mathbb{C}$. Equivalently, $R_2(K)$ is the uniform closure on K of functions $f(\zeta) + \bar{z}(\zeta)$, with f, g being rational functions with poles off K, i.e., $f, g \in R(K)$.

The bi-analytic rational module R_2, and more generally $R_N(K)$ generated by $f_1(\zeta) + \bar{z}f_2(\zeta) + \cdots + \bar{z}^{N-1}f_N(\zeta), f_j \in R(K)$ have been studied intensely in the '70s and '80s — cf., e.g., [18, 16, 17, 19, 14, 15]. The subject remained dormant after that for almost two decades until a remarkable result of Masalov [13] extending Mergelyan's approximation theorem to the rational modules setting.

Here, we want to suggest a different point of view on the approximation by bi-analytic functions based on extending the notion of "analytic content" in [6, 2] to this setting. Namely, let us accept the following definition:

Definition 1. *Let* $\lambda_2(K) := \inf\limits_{\varphi \in R_2(K)} \left\| \frac{\bar{z}^2}{2} - \varphi \right\|_{C(K)}$*, and call* λ_2 *the bi-analytic content of* K*.*

(From now on, $\|\cdot\| = \|\cdot\|_{C(K)}$ unless otherwise specified.) The analogy with $\lambda(K)$, the analytic content defined first in [8], is clear. Indeed, $\lambda(K) := \inf\limits_{\varphi \in R(K)} \|\bar{z} - \varphi(z)\|$, $R(K) = \overline{\text{Ker } \bar{\partial}_{\|\cdot\|}}$ while $\frac{\partial}{\partial \bar{z}}(\bar{z}) = 1$, making $\bar{z}$ the simplest non-analytic function. $R_2(K) = \overline{\text{Ker } \bar{\partial}^2_{\|\cdot\|}}$ and $\bar{\partial}^2 \left(\frac{\bar{z}^2}{2}\right) = 1$.

2 An "Analogue" of the Stone–Weierstrass theorem

The following simple proposition supports the introduction of $\lambda_2(K)$ — cf. [9, 11].

Proposition 1. $\lambda_2(K) = 0$ *iff* $R_2(K) = C(K)$.

Proof (Sketch). The necessity is obvious. To see the sufficiency, note that $\lambda_2(K) = 0$ yields $\bar{z}^2 \in R_2(K)$. Hence one can approximate $\bar{z}^2$ by functions $r_1(z) + \bar{z}\, r_2$. Thus, for any $r \in R_2(K)$ we have $r\bar{z}^2 \sim rr_1 + \bar{z}rr_2$, where we put "$\sim$" for "approximate uniformly". Hence, $\bar{z}^3 \sim \bar{z}\,(r_1 + \bar{z}\, r_2) \sim \bar{z}r_1 + (r_3 + r_4\, \bar{z}) \in R_2(K)$, where all $r_j \in R(K)$. Hence, $\bar{z}^3 \sim r_5 + \bar{z}r_6$ and then $\bar{z}^2\,(r_7 + \bar{z}\, r_8) \in R_2(K)$ since $\bar{z}^3 r_8 \sim (r_5 + \bar{z}\, r_6)\, r_8 \in R_2(K)$. A straightforward induction yields that all monomials $\bar{z}^n z^m$ are approximable by $R_2(K)$. Weierstrass' approximation theorem finishes the argument.

3 Green's Formula and Duality

As is well-known, the fundamental solution for $\bar{\partial}^2$ is $-\frac{1}{\pi}\,\frac{\bar{z}}{z}$. Hence, Green's formula yields immediately the following (cf. [16]).

Lemma 1. *For any $\varphi \in C_0^\infty$ and any $z \in \mathbb{C}$, we have*

$$\varphi(z) = -\int_{\mathbb{C}} \frac{\partial^2 \varphi(\zeta)}{\partial \bar{\zeta}^2}\, \frac{\overline{\zeta - z}}{\zeta - z}\, dA(\zeta). \tag{1}$$

(Here and onward, $dA(\zeta)$ denotes the normalized area measure $\frac{1}{\pi}\, dx\, dy$.)

Lemma 2.

$$\lambda_2(K) = \max_{z \in K} \left| \int_K \frac{\overline{\zeta - z}}{\zeta - z}\, dA(\zeta) \right|. \tag{2}$$

Proof. (A sketch, since the argument is standard, cf., e.g., [6].) Extend $\frac{1}{2}\,\bar{z}^2$ to $\varphi_0 \in C_0^\infty$ with the support in a fixed disk $D = \{z : |z| \leq R < \infty\}$. For $\epsilon > 0$, let Ω_ϵ be a smoothly bounded ϵ-neighborhood of K. For $z \in K$, split the integral in (1) into three parts:

$$\int_{D \setminus \Omega_\epsilon} + \int_{\Omega_\epsilon \setminus K} + \int_K =: I + II + III.$$

$I \in R_2(K)$, $\|II\| \leq const(\varphi_0)\mathrm{Area}(\Omega_\epsilon \setminus K)$, and the statement follows since $\bar{\partial}_z^2 \varphi_0 \equiv 1$ on K.

Set

$$F(z) := \int_K \frac{\overline{\zeta - z}}{\zeta - z}\, dA(\zeta). \tag{3}$$

Clearly, $F(z) \in C^1\left(\mathbb{R}^2\right)$. Thus, $\max_{z \in K} |F(z)|$ occurs somewhere on K.

Let $R_a = R_a(K) = \sqrt{\frac{\mathrm{Area}(K)}{\pi}}$ denote the radius of a disk with the same area as K.

Lemma 3. *$\lambda_2(K) \leq R_a^2$. Moreover,* $\sup\{\lambda_2(K) : \mathrm{Area}(K)$ *is fixed*$\} = R_a^2$*, although there is no "extremal" set K for which equality occurs.*

Proof. The first statement follows from Lemma 2 since the integrand in (3) is bounded by 1. The rest follows at once if one considers a sequence of "cigar-shaped" domains Ω_n with a fixed area symmetric with respect to the x-axis and tangent to the y-axis at the origin. Then, $F(0) \to R_a^2$ since $\frac{\bar\zeta}{\zeta} \to 1$ pointwise on Ω_n and is bounded by 1, so the Lebesgue bounded convergence theorem applies.

Remark 1. Recall that unlike the bi-analytic content, the analytic content ($\lambda(K) := \mathrm{dist}_{C(K)}(\bar z, R(K))$ is bounded above by R_a and the equality holds for disks and only for disks modulo sets of area zero(cf. [2, 6]).

4 Bi-analytic Content of Disks

Proposition 2. *Let $\overline{D} = \{z : |z| \leq R\}$. Then, $\lambda_2\left(\overline{D}\right) = \frac{1}{2} R^2$.*

Proof. By taking $\varphi \equiv 0 \subset R_2\left(\overline{D}\right)$, we see that $\lambda_2\left(\overline{D}\right) \leq \frac{1}{2} R^2$. To obtain the converse inequality, note that for any polynomials P_1, P_2 we have

$$\begin{aligned} \left\| \frac{1}{2}\,\bar z^2 - P_1 - \bar z\, P_2 \right\|_D &\geq \left\| \frac{1}{2}\,\bar z^2 - P_1 - \bar z\, P_2 \right\|_{\partial D} \\ &\geq \inf_{P_1,P_2} \frac{1}{R^2} \left\| \frac{1}{2}\, R^4 - z^2 P_1 - P_2 R^2 z \right\|_{\partial D} \\ &= \inf_{P: P(0)=0} \left\| \frac{1}{2}\, R^2 - P(z) \right\|_{\partial D} = \frac{1}{2}\, R^2. \end{aligned} \tag{4}$$

(The latter infimum is a trivial extremal problem in $H^\infty(D)$-setting (cf. [3], [12, Ch. 8]) and is easily computable, e.g., by duality:

$$\inf_{\substack{f \in H^\infty(D) \\ f(0)=0}} \|C - f\|_{\partial D} = C \sup_{\substack{f \in H^1(D) \\ \|f\|_{H^1}=1}} \left| \int_{\partial D} f\, ds \right| = C,$$

for any constant $C > 0$.) Since P_1, P_2 were arbitrary, the proposition follows.

5 Bounds for $\boldsymbol{\lambda_2}$

The following statement is obvious.

Corollary 1. *Let K be a compact subset of $\mathbb{C}$ and the outer and inner radii R_o, R_i denote, respectively, the minimal radius of a disk containing K (i. e., R_o), and the maximal radius of a disk contained in K. Then,*

$$\frac{1}{2}\, R_i^2 \leq \lambda_2(K) \leq R_o^2. \tag{5}$$

(Of course, here, we tacitly used Runge's theorem in its simplest form: $R\left(\overline{D}\right)$ = uniform closure of polynomials, for any disk D.)

Corollary 2 ([16]). *$R_2(K) = C(K)$ if and only if K is nowhere dense.*

The necessity follows at once from the lower bound in (5) and Proposition 1. The proof of sufficiency, given by Trent and Wang in [16], cannot be shortened or simplified any further. Thus, for the reader's convenience, we only indicate the outline.

(i) By the Hahn–Banach theorem it suffices to check that μ annihilating $R_2(K)$ must be zero, i.e., annihilates all C_0^∞-functions.
(ii) By Lemma 1 and Fubini's theorem, it suffices to check that an R_2-analogue of the Cauchy transform for μ

$$\check{\mu}(z) := \int_{\mathbb{C}} \frac{\overline{\zeta - z}}{\zeta - z}\, d\mu(\zeta) \tag{6}$$

vanishes a.e. wrt dA.
(iii) The Lebesgue bounded convergence theorem yields that $\check{\mu}$ is continuous in $\mathbb{C}$ except at atoms of μ, i.e., at at most countably many points.
(iv) If K is nowhere dense, $\check{\mu}$ vanishes in $\mathbb{C} \setminus K$, and by (iii) in all of $\mathbb{C}$ except for a countable set and the proof is finished.

6 Concluding Remarks

(i) Undoubtedly, the above scheme can be extended to more general "rational modules" associated with the operator $\overline{\partial_z}^N$, i.e., to $R_N(K)$.
(ii) Most likely, one may consider the bi-analytic content or, more generally, N-analytic content for other norms than the uniform norm, e.g., Bergman L^p-norms, Hardy norms, etc. The recent results in that direction for the analytic content ([7, 4, 5]) yield some interesting connections and the latter continue forthcoming.
(iii) It would be interesting to tighten the inequality (5), perhaps obtaining sharper bounds that might involve deeper geometric characteristics of K, e.g., perimeter, capacity, torsional rigidity. For the analytic content this line of inquiry proved to be quite fruitful (cf. [7, 4, 5], cited above).

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